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TECHNICAL TRANSLATION

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THE EFFECT OF THE BEHAVIOR OF THE LOAD ON THE

FREQUENCY OF THE FREE VIBRATIONS OF A RING

By E. B. Wasserman

Translated from "Problems of Dynamics and of Dynamical Strength." Published by the Academy of Sciences of the Latvian SSR (Riga), issue 4, 1956.

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Introduction

In this paper we will consider the plane and three-dimensional free vibrations of elastic circular rings subjected to a uniformly distributed radial load, and, in passing we will also consider the question of their stability.

It is known that the behavior of a load in the process of deformation has a marked influence on the value of the critical load. It is also to be expected that the behavior of the load should have a serious effect on the value of the natural frequencies. Therefore, the present study deals with free vibrations for various cases of load behavior, namely, for the following three possible cases (Fig. 1) frequently encountered in studies on stability, (see /1, 2, 3, 4/ and others):

1) the load remains normal to the deflected axis of the bar;
2) the load remains normal to the undeflected axis of the bar, i.e., remains parallel to its initial direction;
3) the load remains directed toward the initial center of curvature of the bar.

The first case occurs in the pressure of liquids or gases and refers to a hydrostatic load. The force due to weight illustrates the second case. The third case can occur for a wheel with many thin spokes when they are under tension.

For denoting these cases in the plane or two-dimensional problem, we shall use the Arabic numbers 1,2,3 ("Case 1," "Case 2," and "Case 3"); the same cases in the three-dimensional problem will be denoted by the Roman numerals I, II, III ("Case I," "Case II" and "Case III").

The fundamental differential equations for the vibrations of an unloaded circular bar were obtained by Lamb in 1888 /5/; the equations for

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*Translated from "Problems of Dynamics and of Dynamical Strength." Published by the Academy of Sciences of the Latvian SSR (Riga), issue 4, 1956, pp. 49-71.

NASA reviewer's note: Several obviously typographical errors in equations in the original Russian text have been corrected by the reviewer without comment.

a radially loaded circular bar for one case of plane vibrations (Case 1) and two cases of three-dimensional vibrations (Cases I and III) were obtained by K. Federhofer in 1933 /6/.

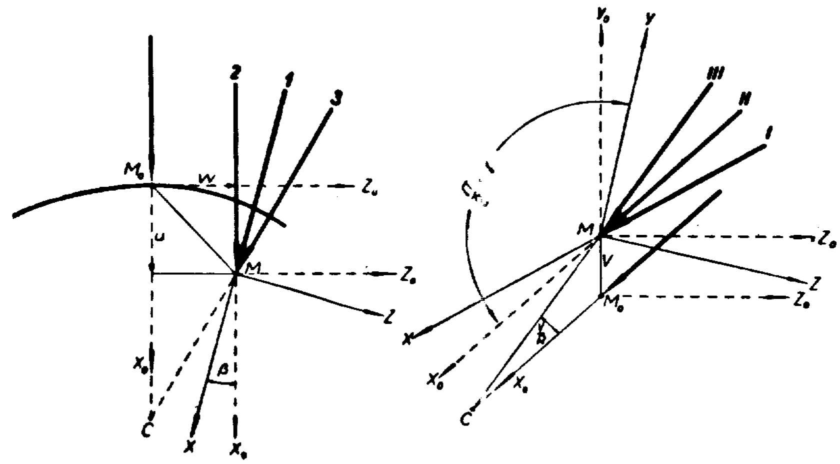

Fig. 1. The three cases of the load behavior.

In the present paper, similar equations are derived for Cases 2 and 3 of plane deformation and Case II of three-dimensional deformation. In the derivations we shall not take into account the effect of a transverse force, the inertia of rotation, and the change in length of the axis in the process of vibration. Moreover, we shall consider the dimensions of the cross-section of the ring small in comparison with its radius.

The three differential equations of vibration obtained and the three Federhofer equations will be employed in this paper for investigating the free plane and three-dimensional vibrations of a ring; in the author's dissertation /10/ (submitted March 26, 1956), these equations were used for investigating the vibrations and stability of circular arcs.

1. FUNDAMENTAL DIFFERENTIAL EQUATIONS OF PLANE AND THREE-DIMENSIONAL VIBRATIONS OF A RADIALLY LOADED CIRCULAR BAR

1.1 Projections of the Distributed Load Vector on the Coordinate Axes

We shall denote by the letter M_0 any point on the axis of the rod before deformation (Fig. 1) and take the origin of a left-handed system of rectangular coordinates x_0, y_0, z_0 to coincide with this point. We shall restrict ourselves to the case where one of the principal central axes of the cross-section lies in the plane of curvature of the bar. The z-axis is taken tangent to the axis of the bar in the direction of increasing arc length s; the x_0- and y_0-axes are taken along the principal central axes of inertia of the cross-section, the plane x_0z_0 coinciding with the plane of the bar. The x_0-axis will then be directed along the principal normal toward the center of the rod and the y_0-axis perpendicular to the plane of the rod (binormal).

Simultaneously with these, we shall consider a left-handed system of rectangular coordinates x, y, z connected with the axis of the bar after its deformation. As the origin we shall take the point M to which the point M_0 has gone over on the deformation of the rod. The z-axis is taken along the tangent to the axis of the bar in the direction of increasing arc; the $\underline{x}$-axis is taken perpendicular to the $\underline{z}$-axis, passing the plane through axis $\underline{z}'$ and tangent to the line into which the $\underline{x}_0$-axis has passed after the deformation; the $\underline{y}$-axis is directed normal to the $\underline{xz}$-plane.

We also denote

$\underline{p}$, the vector of the distributed load;

$\underline{p}_x$, $\underline{p}_y$, $\underline{p}_z$, its projections on the axes $\underline{x}$, $\underline{y}$, $\underline{z}$;

R, the radius of the axis of the bar;

$\underline{u}$,$\underline{v}$,$\underline{w}$, the projections of the displacement vector of a point on the axes $\underline{x}_0$, $\underline{y}_0$, $\underline{z}_0$;

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α, β, γ, the projections of the vector of rotation of the section on the axes $\underline{x}_0$, $\underline{y}_0$, $\underline{z}_0$.

Considering only small displacements, the following relations can be derived for the various cases of behavior of the load in the process of deformation (see /4/ p. 654):

$$\left.\begin{array}{ll} 1. & p_x = p = \text{const} \quad p_z = 0 \\ 2. & p_x = p = \text{const} \quad p_z = p\beta \\ 3. & p_x = p = \text{const} \quad p_z = p\left(\beta - \frac{w}{R}\right) \end{array}\right\} \qquad (1.01)$$

$$\left.\begin{array}{rl} \text{I.} & p_y = 0 \\ \text{II.} & p_y = -p\gamma \\ \text{III.} & p_y = -p\left(\gamma + \frac{v}{R}\right) \end{array}\right\} \qquad (1.02)$$

1.2 Equations of Motion of an Arc Element

Let us consider an element of a circular arc $\underline{ds}$ radially loaded by an external force $\underline{pds}$ (Fig. 2). Setting up the equations of equilibrium and ignoring quantities of the second order of magnitude, we obtain the well-known equations of Kirchhoff for a thin curvilinear bar.

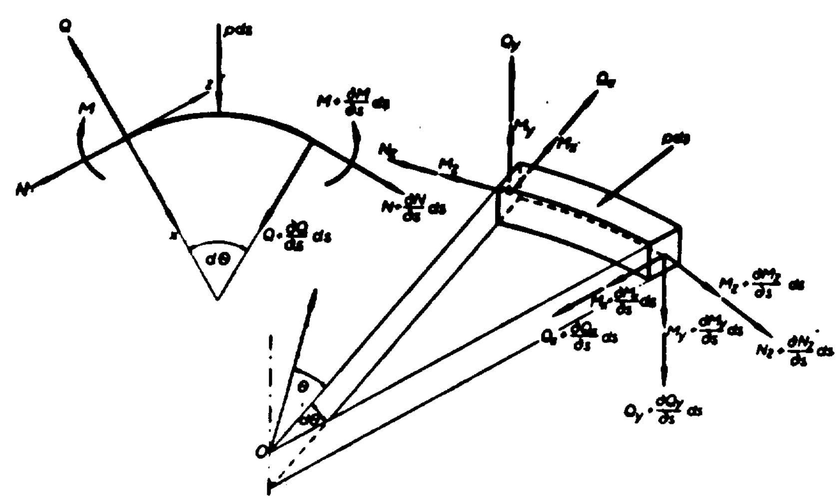

Fig. 2. Forces and moments acting on an element of a ring.

Using the kinematic relations derived by Clebsch

$$\alpha = -\frac{1}{R} v^{\mathrm{I}} \tag{1.03}$$

$$\beta = \frac{1}{R}(u^{\mathrm{I}} + w), \tag{1.04}$$

and taking into account the inertial forces (ignoring the inertia of rotation), we obtain a system of six equations:

$$Q_x^{\mathrm{I}} + (1 + R\delta q) N_z + Rp_x - mR\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.05}$$

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$$N_z^{\mathrm{I}} - Q_x + Rp_z - mR\frac{\partial^2 w}{\partial t^2} = 0 \tag{1.06}$$

$$M_y^{\mathrm{I}} + RQ_x = 0 \tag{1.07}$$

$$Q_y^{\mathrm{I}} - R\delta p N_z + Rp_y - mR\frac{\partial^2 v}{\partial t^2} = 0 \tag{1.08}$$

$$M_x^{\mathrm{I}} + M_z - RQ_y = 0 \tag{1.09}$$

$$M_z^{\mathrm{I}} - M_x = 0 \tag{1.10}$$

The Roman numbers here denote the partial derivatives with respect to the angle Θ (for example),

$$M_z^{\mathrm{I}} = \frac{\partial M_z}{\partial \Theta}\Big),$$

where Θ is the variable central angle measured from the axis of symmetry in the clockwise direction;
$\underline{m}$, the mass of a unit length of the bar;
$\underline{t}$, the time;
$\underline{N_z}$, the normal force;
$\underline{Q_x}$ and $\underline{Q_y}$ the transverse forces;
$\underline{M_x}$ and $\underline{M_y}$, the bending moments; and
δp, δq, δr, the increments of the principal components of the curvature and torsion of the bar in deformation.
In the derivation it was taken into account that $\underline{ds} = \underline{Rd}\,\Theta$.
In subsequent derivations we express the increments of the principal components of curvature and torsion and the elastic moments in terms of the angles of turning and displacement.

$$\delta p = \frac{1}{R}\left(-\frac{1}{R} v^{\mathrm{II}} + \gamma\right) \tag{1.11}$$

$$\delta q = \frac{1}{R}\beta^{\mathrm{I}} \tag{1.12}$$

$$\delta r = \frac{1}{R}\left(\gamma^{\mathrm{I}} + \frac{1}{R} v^{\mathrm{I}}\right) \tag{1.13}$$

$$M_x = \frac{B_x}{R}\left(-\frac{1}{R} v^{\mathrm{II}} + \gamma\right) \tag{1.14}$$

$$M_y = \frac{B_y}{R^2}(u^{\mathrm{II}} + u) \tag{1.15}$$

$$M_z = \frac{C}{R}\left(\gamma^{\mathrm{I}} + \frac{1}{R} V^{\mathrm{I}}\right) \tag{1.16}$$

Here $\underline{B_x} = \underline{EI_x}$ and $\underline{B_y} = \underline{EI_y}$ are the principal flexural rigidities, and $\underline{C} = \underline{GI_k}$ the torsional rigidity

Moreover, we make use of the condition of the noncompressibility of the axis, which leads to the relation

$$u = w^{\mathrm{I}}. \tag{1.17}$$

1.3 The Plane Problem

The plane problem is defined by Equations (1.05) - (1.07). We transform Eq. (1.05), representing the normal force in the vibration process, as the sum:

$$N_z = -pR + N, \tag{1.18}$$

where $\underline{N}$ is the increase in the normal force due to vibration.
Taking Eq. (1.12) into account we obtain in place of (1.05) - (1.07):

$$Q_x^{\mathrm{I}} - pR + N - pR\beta^{\mathrm{I}} + Rp_x - mR\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.19}$$ *

$$N^{\mathrm{I}} - Q_x + Rp_z - mR\frac{\partial^2 w}{\partial t^2} = 0 \tag{1.20}$$

$$M_y^{\mathrm{I}} + RQ_x = 0 \tag{1.07}$$

Eliminating the force N from Eq. (1.19) and (1.20) we have:

$$Q_x^{\mathrm{II}} + Q_x - pR\beta^{\mathrm{II}} + R(p_x^{\mathrm{I}} - p_z) + mR\frac{\partial^2}{\partial t^2}(w - u^{\mathrm{I}}) = 0 \tag{1.21}$$

From Eqs. (1.07) and (1.15) we obtain:

$$Q_x = -\frac{B_y}{R^3}(u^{\mathrm{III}} + w^{\mathrm{II}}). \tag{1.22}$$

Substituting Eq. (1.22) into (1.21) and making use of Eqs. (1.04) and (1.17), we obtain the fundamental differential equation for the problem of the general case of the behavior of a load.

$$\begin{aligned} &w^{\mathrm{VI}} + \left(2 + \frac{pR^3}{B_y}\right) w^{\mathrm{IV}} + \left(1 + \frac{pR^3}{B_y}\right) w^{\mathrm{II}} + \\ &+ \frac{mR^4}{B_y}\frac{\partial^2}{\partial t^2}(w^{\mathrm{II}} - w) + \frac{R^4}{B_y}(p_z - p_x^{\mathrm{I}}) = 0. \end{aligned} \tag{1.23}$$

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*NASA reviewer's note: The omission of the term $N\beta^{\mathrm{I}}$ in equation (1.19) implies that the subsequent development is applicable only to cases for which inertia loads are small compared to the applied loads ($N \ll pR$); that is, the development is generally inappropriate for high frequency vibrations.

Substituting relations (1.01) into (1.23), we obtain the following equations for the vibrations:

for Case 2:

$$w^{VI} + (2 + q_2)\, w^{IV} + (1 + 2q_2)\, w^{II} + q_2 w + \frac{mR^4}{B_y}\,\frac{\partial^2}{\partial t^2}\,(w^{II} - w) = 0; \qquad (1.24)$$

for Case 3:

$$w^{VI} + (2 + q_3)\, w^{IV} + (1 + 2q_3)\, w^{II} + \frac{mR^4}{B_y}\,\frac{\partial^2}{\partial t^2}\,(w^{II} - w) = 0. \qquad (1.25)$$

where

$$q_2 = \frac{p_2 R^3}{B_y}, \quad q_3 = \frac{p_3 R^3}{B_y}. \qquad (1.26)$$

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We shall henceforth denote the dimensionless magnitude q as the load parameter.

1.4 The Three-Dimensional Problem

We shall derive the differential equations of vibration for Case II (Cases I and III have been considered by Federhofer). Transforming Eq. (1.08), taking into account Eqs. (1.18), (1.02), and (1.11), we obtain:

$$Q_y^{I} - pv^{II} - mR\frac{\partial^2 v}{\partial t^2} = 0. \qquad (1.27)^{*}$$

Differentiating Eq. (1.09) with respect to θ and substituting the values of Q_y^{I} and M_z^{I} from Eqs. (1.27) and (1.10), we have an equation containing two unknown functions, M_x and v.

$$M_x^{II} + M_x - pRv^{II} - mR^2\frac{\partial^2 v}{\partial t^2} = 0. \qquad (1.28)$$

To obtain one more equation, we solve Eq. (1.14) for γ and substitute in the expression obtained into Eq. (1.16):

$$M_z = \frac{C}{R}\left[\left(\frac{R}{B_x}M_x^{I} + \frac{1}{R}\, v^{III}\right) + \frac{1}{R}\, v^{I}\right]. \qquad (1.29)$$

In order to eliminate the magnitude M_x from Eq. (1.28) and (1.29), we differentiate Eq. (1.29) with respect to θ and, taking into account Eq. (1.10), obtain:

$$-M_x^{II} + M_x\frac{B_x}{C} - \frac{B_x}{R^2}(v^{IV} + v^{II}) = 0. \qquad (1.30)$$

This equation is valid for all three cases of load behavior under consideration.

Adding Eqs. (1.28) and (1.30), we have:

$$M_x = \frac{pRv^{II} + mR^2\frac{\partial^2 v}{\partial t^2} + \frac{B_x}{R^2}(v^{IV} + v^{II})}{1 + \frac{B_x}{C}}. \qquad (1.31)$$

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*NASA reviewer's note: The frequency restrictions mentioned in the footnote on page 5 apply also to equation (1.27).

Substituting Eq. (1.31) into Eq. II.28), we obtain , after simplification, an equation containing only one unknown function, v:

$$v^{\mathrm{VI}}+(2+q_{\mathrm{II}})v^{\mathrm{IV}}+(1-\lambda q_{\mathrm{II}})v^{\mathrm{II}}+$$
$$+mR^4\left(\frac{1}{B_x}\frac{\partial^2}{\partial t^2}v^{\mathrm{II}}-\frac{1}{C}\frac{\partial^2 v}{\partial t^2}\right)=0. \tag{1.32}$$

where

$$q_{\mathrm{II}}=\frac{p_{\mathrm{II}}R^3}{B_x}, \tag{1.33}$$
$$\lambda=\frac{B_x}{C}. \tag{1.34}$$

We shall call the magnitude λ the rigidity ratio.

1.5 General Solution of the Fundamental Differential Equations.
We shall demonstrate the method of solution of the fundamental differential equations using as an example Eq. (1.24). The remaining equations are solved analogously. We seek the solution in the form of the product of two functions, one of which depends only on Θ, the other only on t:

$$w=\overline{w}\cdot f(t). \tag{1.35}$$

In what follows the bar will denote functions depending only on the angle Θ.
Substitution of Eq. (1.35) into Eq. (1.24) **after division by f(t), yields:**

$$\frac{\overline{w}^{\mathrm{VI}}+(2+q_2)\overline{w}^{\mathrm{IV}}+(1+2q_2)\overline{w}^{\mathrm{II}}+q_2\overline{w}}{\frac{mR^4}{B_y}(\overline{w}^{\mathrm{II}}-\overline{w})}=-\frac{f''(t)}{f(t)}. \tag{1.36}$$

Satisfying Eq. (1.36) (for any Θ and t) requires **that each side be equal to the same constant,** which we shall denote by ω^2.
We then obtain the two equations:

$$-\frac{f''(t)}{f(t)}=\omega^2, \tag{1.37}$$

$$\frac{\overline{w}^{\mathrm{VI}}+(2+q_2)\overline{w}^{\mathrm{IV}}+(1+2q_2)\overline{w}^{\mathrm{II}}+q_2\overline{w}}{\frac{R^4m}{B_y}(\overline{w}^{\mathrm{II}}-\overline{w})}=\omega^2. \tag{1.38}$$

As we know, Eq. (1.37) has the solution:

$$f(t)=a\sin(\omega t+\alpha). \tag{1.39}$$

It can now be seen immediately that the magnitude ω is the frequency of the vibrations. Equation (1.38) determines the forms of vibration of the bar.
For the cases we are considering, the corresponding equations (including the three equations derived by Federhofer /6/ for Cases 1, I, and III can be written in the following form:

Case 1

$$\overline{w}^{VI} + (2 + q_1)\overline{w}^{IV} + (1 + q_1 - f_1)\overline{w}^{II} + f_1\overline{w} = 0. \qquad (1.40)$$

Case 2 (directly from Eq. (1.38)

$$\overline{w}^{VI} + (2 + q_2)\overline{w}^{IV} + (1 + 2q_2 - f_2)\overline{w}^{II} + (q_2 + f_2)\overline{w} = 0. \qquad (1.41)$$

Case 3

$$\overline{w}^{VI} + (2 + q_3)\overline{w}^{IV} + (1 + 2q_3 - f_3)\overline{w}^{II} + f_3\overline{w} = 0. \qquad (1.42)$$

Case I

$$v^{VI} + (2 + q_I)\, v^{IV} + (1 + q_I - f_I)\, v^{II} + \lambda f_I v = 0. \qquad (1.43)$$

Case II

$$\overline{v}^{VI} + (2 + q_{II})\overline{v}^{IV} + (1 - \lambda q_{II} - f_{II})\overline{v}^{II} + \lambda f_{II}\overline{v} = 0. \qquad (1.44)$$

Case III

$$\begin{aligned}\overline{v}^{VI} + (2 + q_{III})\overline{v}^{IV} + (1 + q_{III} - \lambda q_{III} - f_{III})\overline{v}^{II} + \\ + \lambda (f_{III} - q_{III})\overline{v} = 0.\end{aligned} \qquad (1.45)$$

Here

$$f_1 = \frac{mR^4}{B_y}\omega_1^2; \quad f_2 = \frac{mR^4}{B_y}\omega_2^2; \quad f_3 = \frac{mR^4}{B_y}\omega_3^2; \qquad (1.46)$$

$$f_I = \frac{mR^4}{B_x}\omega_I^2; \quad f_{II} = \frac{mR^4}{B_x}\omega_{II}^2 \quad f_{III} = \frac{mR^4}{B_x}\omega_{III}^2. \qquad (1.47)$$

We shall henceforth denote the dimensionless magnitude <u>f</u> as the frequency parameter.

By analogy with Eq. (1.26), we set $q_1 = \frac{p_1 R^3}{B_y}$, and by analogy with Eq. (1.33), $q_I = \frac{p_I R^3}{B_x}$, $q_{III} = \frac{p_{III} R^3}{B_x}$.

The particular integral of Eqs. (1.40) - (1.45) we take in the form

$$a \sin (n\Theta + \alpha).$$

We then obtain the following characteristic equations:

Case 1

$$n^6 - (2 + q_1)n^4 + (1 + q_1 - f_1)n^2 - f_1 = 0, \qquad (1.48)$$

Case 2

$$n^6 - (2 + q_2)n^4 + (1 + 2q_2 - f_2)n^2 - (f_2 + q_2) = 0. \qquad (1.49)$$

Case 3

$$n^6 - (2 + q_3)\,n^4 + (1 + 2q_3 - f_3)\,n^2 - f_3 = 0. \qquad (1.50)$$

Case I

$$n^6 - (2 + q_{\mathrm{I}})\,n^4 + (1 + q_{\mathrm{I}} - f_{\mathrm{I}})\,n^2 - \lambda f_{\mathrm{I}} = 0. \qquad (1.51)$$

Case II

$$n^6 - (2 + q_{\mathrm{II}})\,n^4 + (1 - \lambda q_{\mathrm{II}} - f_{\mathrm{II}})\,n^2 - \lambda f_{\mathrm{II}} = 0. \qquad (1.52)$$

Case III

$$n^6 - (2 + q_{\mathrm{III}})\,n^4 + (1 + q_{\mathrm{III}} - \lambda q_{\mathrm{III}} - f_{\mathrm{III}})\,n^2 - \lambda f_{\mathrm{III}} = 0. \qquad (1.53)$$

The general integral of the fundamental differential equations (1.40) - (1.42) has the form:

$$\overline{w} = \sum_{k=1}^{3} (A_k \cos n_k\Theta + B_k \sin n_k\Theta), \qquad (1.54)$$

where $\pm n_1$, $\pm n_2$, $\pm n_3$ are the roots of che corresponding characteristic equations, and A_1, A_2, A_3, B_1, B_2, B_3 are the integration constants. The general integral of Eqs. (1.43) - (1.45) for v has the same form.

2. VIBRATIONS AND STABILITY OF THE RING

2.1 Determination of the Natural Frequencies of the Vibrations

For a ring, the frequency parameters are easily determined directly from the characteristic equations (1.48) - (1.53). Since the magnitude n in these equations represents the number of waves for one passage aroung the ring, which is closed, this number must be an integer ($n = 2,3,4,\ldots$).

The frequency parameters are therefore:

Case 1

$$f_1 = \frac{n^2 (n^2 - 1)(n^2 - 1 - q_1)}{n^2 + 1}. \qquad (2.01)$$

Case 2

$$f_2 = \frac{(n^2 - 1)^2 (n^2 - q_2)}{n^2 + 1}. \qquad (2.02)$$

Case 3

$$f_3 = \frac{n^2 [(n^2 - 1)^2 - (n^2 - 2)\, q_3]}{n^2 + 1}. \qquad (2.03)$$

Case I

$$f_{\mathrm{I}} = \frac{n^2 (n^2 - 1)(n^2 - 1 - q_{\mathrm{I}})}{n^2 + \lambda}. \qquad (2.04)$$

while the vertical component, on the contrary, acts to remove the half-ring from its initial position. From this point of view, the half-ring is most easily deformed in Case II, where there is no restoring horizontal force, and it is deformed with most difficulty in Case I, where the restoring horizontal force is relatively larger than in the other cases. It is well known that the more flexible the system, the more easily it is deformed and the lower its frequency.

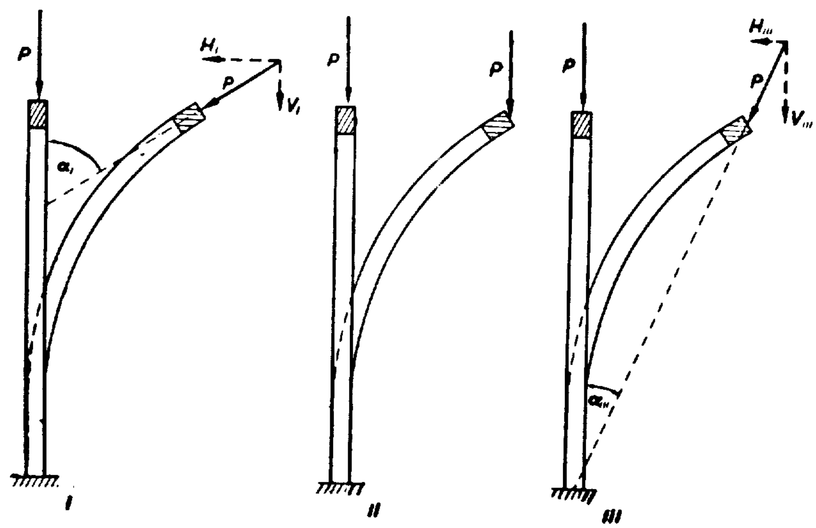


Fig. 3. Effect of load behavior on the yielding of a half-ring.

It is natural, therefore, that under otherwise equal conditions Case II has the lowest frequency and Case I, the highest.

2.2 Determination of the Critical Loads

A compressive load naturally lowers the frequency of the free vibrations and reduces it to zero when the load attains a critical value. On this basis it is easy to determine the values of q, setting $f = 0$ in Eqs. (2.01) - (2.06).

We denote by q_{cr} that value of the dimensionless load parameter for which the load attains the critical value.

$$q_{cr}^{pl} = \frac{p_{cr}^{pl} R^3}{B_y}, \qquad (2.13)$$

$$q_{cr}^{sp} = \frac{p_{cr}^{sp} R^3}{B_x}. \qquad (2.14)$$

This value is called the coefficient of stability. Equation (2.13) refers to Cases 1, 2, and 3, and Eq. (2.14) to Cases I, II, and III.

From Eqs. (2.01) - (2.06) it is easy to obtain the coefficients of stability for all the cases of load behavior considered.

$$q_{1\,cr} = n^2 - 1; \qquad (2.15)$$

$$q_{2\,cr} = n^2; \qquad (2.16)$$

$$q_{3\,cr} = \frac{(n^2 - 1)^2}{n^2 - 2}; \qquad (2.17)$$

$$q_{I\,cr} = n^2 - 1; \tag{2.18}$$

$$q_{II\,cr} = \frac{(n^2-1)^2}{n^2+\lambda}, \tag{2.19}$$

$$q_{III\,cr} = \frac{n^2(n^2-1)}{n^2+\lambda}. \tag{2.20}$$

Comparing the values obtained for the coefficients of stability, it can easily be seen that

$$q_{1\,cr} < q_{2\,cr} < q_{3\,cr}, \quad \text{i.e.} \quad p_{1\,cr} < p_{2\,cr} < p_{3\,cr},$$

and

$$q_{II\,cr} < q_{III\,cr} < q_{I\,cr}, \quad \text{i.e.} \quad p_{II\,cr} < p_{III\,cr} < p_{I\,cr}.$$

These results agree with the results obtained above for the frequencies; namely, in the plane problem the smallest critical load is hydrostatic, and in the three-dimensional problem it is the load which does not vary its direction in the process of deformation.

From Eqs. (2.15) - (2.20) it follows also that for plane vibrations the difference between the stability coefficients for the second and first cases of load behavior does not depend on the number of waves and is always equal to unity:

$$q_{2\,cr} - q_{1\,cr} = 1.$$

The same difference for the third and second cases of load behavior depends on the number of waves:

$$q_{3\,cr} - q_{2\,cr} = \frac{1}{n^2-2}.$$

2.3 Comparison of the Characteristics of Plane and Three-Dimensional Vibrations of a Ring for Different Cross-Sectional Shapes.

2.3.1 Circular, Tubular, and Square Cross-Section.

For comparison of the characteristics of the vibrations of rings of circular, tubular, and square cross-sections, it is convenient to use a graph showing the dependence of the frequency parameter f on the load parameter q. From Eqs. (2.01) - (2.06) it follows that for a ring this dependence is linear and is represented by a straight line.

Computing the frequency parameters of an unloaded ring from Eqs. (2.07) and (2.08) and the stability coefficients from Eqs. (2.15) - (2.20), taking n = 2 and n = 3, we construct a graph (Fig. 4) for the circular or tubular section for which the rigidity ratio is:

$$\lambda = \frac{B_x}{C} = \frac{EI_x}{G \cdot I_p} = \frac{E}{2G} = 1 + \mu.$$

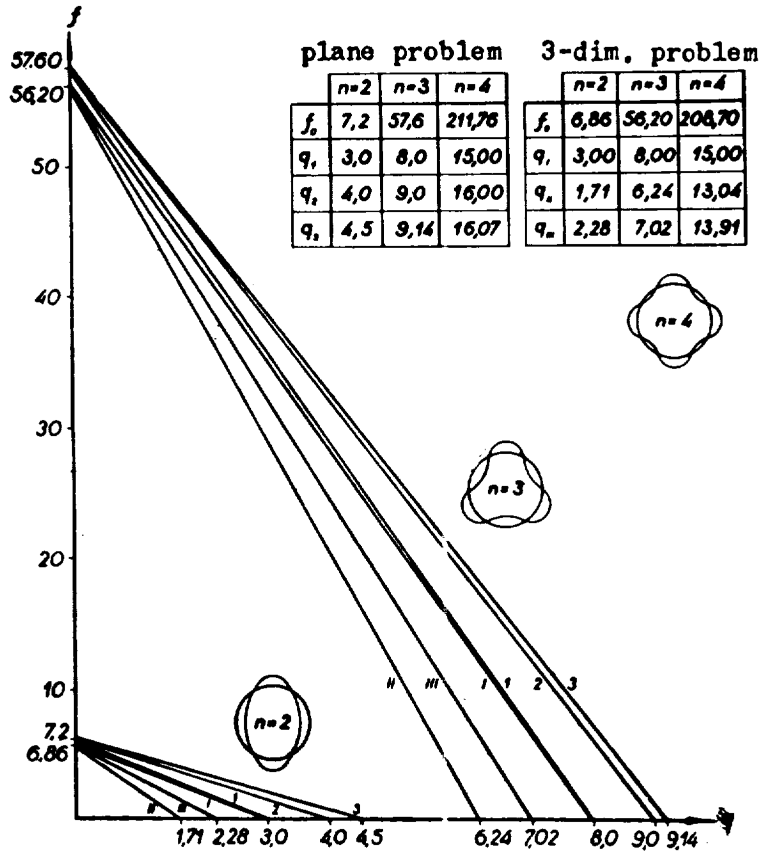
The Poisson coefficient will be taken here and in what follows as

$$\mu = 0.25.$$

Then for the circular and tubular sections, $\lambda = 1{,}25$.

It is seen from the graph that for a ring of circular or tubular section, for any number of waves and any behavior of the load in the process of deformation,

$$f_{sp} < f_{pl}, \quad \text{i.e.} \quad \omega_{sp} < \omega_{pl}.$$



plane problem

	n=2	n=3	n=4
f_0	7,2	57,6	211,76
q_1	3,0	8,0	15,00
q_2	4,0	9,0	16,00
q_3	4,5	9,14	16,07

3-dim. problem

	n=2	n=3	n=4
f_0	6,86	56,20	208,70
q_I	3,00	8,00	15,00
q_{II}	1,71	6,24	13,04
q_{III}	2,28	7,02	13,91

Fig. 4. Vibrations of a ring (circular section).

Thus, in the case considered, it is the three-dimensional form of vibrations that is of practical interest.

The same conclusion can also be drawn as regards a ring of square cross-section, for which the rigidity ratio is:

$$\lambda = \frac{EI_x}{G \cdot I_p} = \frac{\frac{1}{12} Eb^4}{0.4E \cdot 0.141 b^4} = 1.48.$$

For the sections considered, the difference between the frequencies ω_0^{pl} and ω_0^{sp} for an unloaded ring is very small. In fact, according to Eqs. (2.07) and (2.08),

$$\frac{f_0^{pl}}{f_0^{sp}} = \frac{n^2+\lambda}{n^2+1}.$$

Since for the circular and tubular sections λ = 1.25 and for the square section λ = 1.48, we find that even for n = 2, the smallest number of waves,

$$\frac{\omega_0^{pl}}{\omega_0^{sp}} = \sqrt{\frac{n^2+1}{n^2+\lambda}} \approx 1.$$

2.3.2 Rectangular Section

a) Ratio of Frequencies of Plane and Three-Dimensional Vibrations of an Unloaded Ring.

Denote the dimensions of the section in the plane of the ring by b and the dimensions of the section perpendicular to the plane of the ring by h, (Fig. 5).

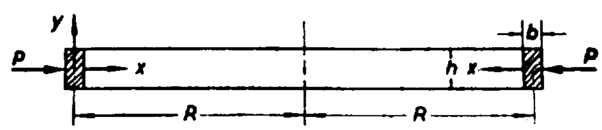


Fig. 5. Cross-Section of Ring.

According to Eqs. (2.09) and (2.10) the ratio of the frequency parameters of an unloaded ring for plane and three-dimensional deformations for n = 2 equals:

$$\frac{f_0^{pl}}{f_0^{sp}} = \frac{4+\lambda}{5}.$$

Taking into account Eqs. (1.46) and (1.47) and also the fact that $I_x = bh^3/12$, $I_y = hb^3/12$, we obtain:

$$\left(\frac{h}{b}\right)^2 \frac{\omega_{pl}^2}{\omega_{sp}^2} = \frac{4+\lambda}{5}. \qquad (2.21)$$

In order to determine for what ratio of the cross-section sides the frequency of the natural three-dimensional vibrations will be equal to the frequency of the natural plane vibrations, we substitute in Eq. (2.21)

$$\omega_{pl} = \omega_{sp}.$$

We then obtain

$$\left(\frac{h}{b}\right)^2 = \frac{4+\lambda}{5}. \qquad (2.22)$$

The value of the stiffness ratio λ as a function of the ratio of the sides h/b is determined for μ = 0.25 from the following equations:

for $\underline{h} > \underline{b}$

$$\lambda = \frac{1}{4.8k'}\left(\frac{h}{b}\right)^2, \qquad (2.23)$$

for $\underline{h} < \underline{b}$

$$\lambda = \frac{1}{4.8k'}, \qquad (2.24)$$

where $\underline{k}'$ is a coefficient depending only on the ratio $\underline{h}/\underline{b}$, (taken from /9/, table on p. 44). For intermediate values of $\underline{h}/\underline{b}$, the coefficient $\underline{k}'$ is determined by linear interpolation. F

Computations show that Eq. (2.22) is satisfied for $\underline{h}/\underline{b} = 1.056$, when 5
$\lambda = 1.573$. 2

For sections for which $\underline{h}/\underline{b} > 1.056$, $\omega_o^{sp} > \omega_o^{pl}$, while for sections for which $\underline{h}/\underline{b} < 1.056$, $\omega_o^{sp} < \omega_o^{pl}$.

b) Comparison of Cases 1 and I.

The ratio of the frequency parameters, in agreement with Eqs. (2.04) and (2.01), for n = 2 equals:

$$\frac{f_I}{f_1} = \frac{5(3-q_1)}{(4+\lambda)(3-q_1)}.$$

Then

$$\frac{\omega_I^2}{\omega_1^2} = \frac{5\left(3\frac{B_x}{B_y} - q_1\right)}{(4+\lambda)(3-q_1)}.$$

Since $B_x/B_y = (h/b)^2$, then in the case in which $\omega_I = \omega_1$, this equation assumes the form

$$(4+\lambda)(3-q_1) = 5\left[3\left(\frac{h}{b}\right)^2 - q_1\right].$$

whence

$$q_1 = \frac{12 + 3\lambda - 15\left(\frac{h}{b}\right)^2}{\lambda - 1}. \qquad (2.25)$$

This equation represents a curve corresponding to the equality of the frequencies for plane and three-dimensional vibrations (for n = 2 and $\mu = 0.25$). This curve is shown in Fig. 6a.

It can be seen that for rings subjected to a hydrostatic load, when $\underline{h}/\underline{b} < 1$ the frequency of the three-dimensional vibrations is lower than the frequency of the two-dimensional vibrations, while for $\underline{h}/\underline{b} > 1.056$ the reverse is true. In the region $1.000 < \underline{h}/\underline{b} < 1.056$ the frequencies of the three-dimensional vibrations may be both lower and higher than the corresponding frequencies of the plane vibrations (depending on the magnitude of the load).

The upper point of the curve determines that ratio of the sides of the cross-section for which the critical loads corresponding to the plane and

F
5
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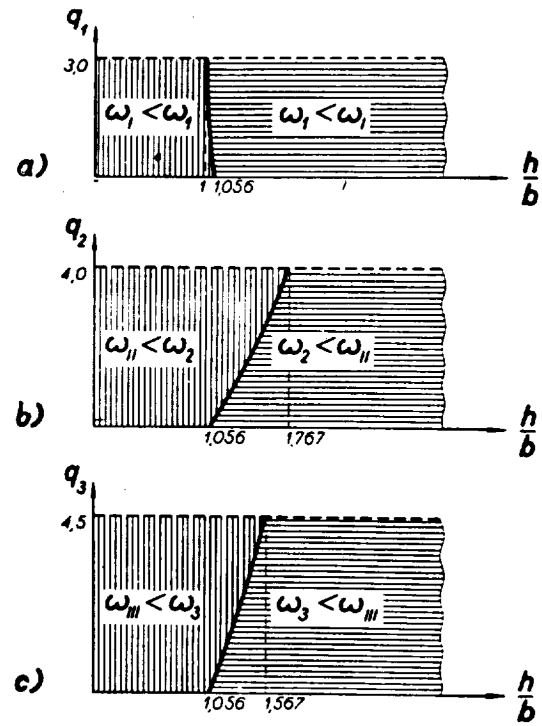

Fig. 6. Frequency regions of three-dimensional and plane vibrations of a ring.

three-dimensional forms of the loss of stability are the same.*

For $h/b < 1$ $\quad p_{Icr} < p_{1cr}$

For $h/b > 1$ $\quad p_{1cr} < p_{Icr}$

c) Comparison of Cases 2 and II.

In agreement with Eqs. (2.05) and (2.02), the ratio of the frequency parameters for n = 2 is:

$$\frac{f_{II}}{f_2} = \frac{20\,[9 - q_{II}\,(4+\lambda)]}{9\,(4+\lambda)\,(4-q_2)}.$$

whence

$$\frac{\omega_{II}^2}{\omega_2^2} = \frac{20\left[9\,\dfrac{B_x}{B_y} - q_2(4+\lambda)\right]}{9\,(4+\lambda)\,(4-q_2)}.$$

Equating ω_{II} with ω_2, we obtain the equation:

$$9(4+\lambda)\,(4-q_2) = 20\left[9\left(\frac{h}{b}\right)^2 - q_2(4+\lambda)\right],$$

* In fact, the coefficients of stability for n = 2, according to Eqs. (2.15) and (2.18), are $q_{1cr} = 3.0$; $q_{Icr} = 3.0$. From Eqs. (2.13) and (2.14) it follows that $p_{1cr} = 3.0\,\frac{B_y}{R^3}$; $\quad p_{Icr} = 3.0\,\frac{B_x}{R^3}$.

The last magnitudes are the same for a square cross-section.

whence it follows that

$$q_2 = \frac{180\left(\frac{h}{b}\right)^2}{11(4+\lambda)} - \frac{36}{11} \qquad (2.26)$$

Equation (2.26) describes the curve of equal frequencies shown in Fig. 6b. As can be seen from the graph, in this case of the behavior of the load when $\underline{h/b} \ll 1.056$, the frequency of the three-dimensional vibrations is always lower than the frequency of the plane vibrations, while when $\underline{h/b} \gg 1.767$, the reverse is true. In the region $1.056 < \underline{h/b} < 1.767$, the frequencies of the three-dimensional vibrations can be both lower and higher than the F
frequencies of the corresponding plane vibrations (depending on the 5
magnitude of the load). 2

Here the top point of the curve shows likewise that for $\underline{h/b} = 1.767$,

$$p_{2\,cr} = p_{II\,cr}.$$

d) Comparison of Cases 3 and III.

According to Eqs. (2.06) and (2.03), the ratio of the frequency parameters for n = 2 equals:

$$\frac{f_{III}}{f_3} = \frac{15[12 - q_{III}(4+\lambda)]}{4(9-2q_3)(4+\lambda)},$$

whence, for the ratio of the squares of the frequencies:

$$\frac{\omega_{III}^2}{\omega_3^2} = \frac{3.75\left[\frac{12B_x}{B_y} - (4+\lambda)q_3\right]}{(4+\lambda)(9-2q_3)}.$$

Taking $\omega_{III} = \omega_3$, we have

$$3.75\left[12\left(\frac{h}{b}\right)^2 - (4+\lambda)q_3\right] = (4+\lambda)(9-2q_3).$$

whence

$$q_3 = \frac{25.714\left(\frac{h}{b}\right)^2}{4+\lambda} - 5.143. \qquad (2.27)$$

From the graph (Fig. 6c), it follows that for rings in the third case of load behavior, when $\underline{h/b} \ll 1.056$, the frequency of the three-dimensional vibrations is always lower than the frequency of the plane vibrations, while for $\underline{h/b} \gg 1.567$, the reverse is true.

In the region $1.056 < \underline{h/b} < 1.567$, the frequency of the three-dimensional vibrations can be both lower and higher than the frequencies of the corresponding plane vibrations, depending on the magnitude of the load.

Analogously, as in the preceding case, it is easy to show that for $\underline{h/b} = 1.567$,

$$p_{3\,cr} = p_{III\,cr}.$$

2.4 <u>Application of the Fundamental Differential Equations of Vibration of a Circular Bar to the Computation of the Free Vibrations and Stability of Arcs</u>

Equations (1.40) - (1.45) were employed by us also for investigating the problem of the free plane and three-dimensional vibrations of loaded flexible nonhinged and two-hinged arcs /10/, both the exact and approximate solutions being obtained.

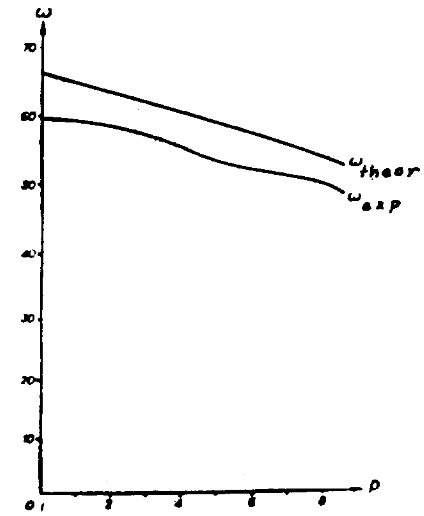

Fig. 7. Experimental and theoretical frequencies of the natural vibrations of a hingeless arc.

From the results obtained, we took for experimental check the case of three-dimensional vibrations of a hingeless arc where the load in the process of deformation remains directed toward the center of the initial form of the arc (Case III).

It was found that the results of the approximate theoretical computation were higher than the experimental results by 7.7 - 11% (Fig. 7).

Two reasons can be given for such a discrepancy.

1. The theoretically computed frequencies were obtained by the Bubnov-Galerkin method, which always gives higher values for the first frequency.

2. The experimentally obtained frequencies are somewhat lower on account of a certain yielding of the clamps.

All this permits us to conclude that the accuracy of the approximate theory is sufficient for practical purposes.

CONCLUSION

In the present paper the problem of the free plane and three-dimensional vibrations of loaded flexible circular rings is investigated. The fundamental results obtained are the following:

1. Differential equations were obtained for a radially loaded circular bar for three cases of the behavior of the load in the process of deformation (Cases 2,3, and II).

2. It was established that the behavior of the load had a considerable effect on the frequency parameter and on the coefficient of stability. It was proven that the following inequality holds for a ring:

$$\omega_{II} < \omega_{III} < \omega_I$$

3. It was established that for rings of circular, tubular, and square cross-sections in any of the three cases of load behavior considered, the n-th natural frequency of the three-dimensional vibrations is always lower than the corresponding n-th frequency of the plane vibrations.

For rings of rectangular section in these cases of load behavior, the regions were determined in which the lowest frequency of the natural vibrations in plane deformation is lower or higher than the corresponding frequencies for three-dimensional deformation.

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